# Wormholes, naked singularities, and universes of ghost radiation

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Both the static and homogeneous metrics describing the spherically symmetric gravitational field of a crossflow of incoming and outgoing null dust streams are generalized for the case of the two-component ghost radiation. Static solutions represent either naked singularities or the wormholes recently found by Hayward. The critical value of the parameter separating the two possibilities is given. The wormhole is allowed to have a positive mass. The homogeneous solutions are open universes.

The static, spherically symmetric solution describing the gravitational field of a crossflow of null dust streams [1] was recently extended by Hayward [2] for the case of two-component ghost radiation, the resulting solution being interpreted as a wormhole.[1] The original static solution [1] however has a homogeneous counterpart [6,7], the similar generalization of which is the principal aim of this Brief Report. We also find the critical value of one of the metric parameters, which separates naked singularities from wormholes in the static case. We show that, despite the negative energy density of the ghost radiation, the wormholes can have a positive mass.

The null dust solutions can be concisely written as

$$ds^2 = \frac{ae^{L^2}}{cR}(dZ^2 - 2R^2 dL^2) + R^2 d\Omega^2, \tag{1}$$

where $c = -1$ refers to the static case (naked singularities) and $c = 1$ refers to the homogeneous case (closed universe). Providing $R > 0$, the metric has the signature $(c, -c, +, +)$. Accordingly, $Z$ is time coordinate in the static case and radial coordinate in the homogeneous case while $L$ is time coordinate in the homogeneous case and radial coordinate in the static case. The parameter's notation was changed to $a = 1/C > 0$ for easier comparison with Ref. [2]. A second parameter $B$ is contained in the metric function $R$,

$$cR = a(e^{L^2} - 2L\Phi_B), \quad \Phi_B = B + \int^L e^{x^2} dx. \tag{2}$$

Both metrics have a true singularity at $R = 0$. ($\mathcal{R}_{ab}\mathcal{R}^{ab} = 2/a^2 e^{2L^2} R^2$ and the Kretschmann scalar $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ contains $R^6$ in the denominator.) The energy-momentum tensor is

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[1] For recent developments in the subject see [3,4] and for an earlier review [5].

$$T_{ab} = \frac{\beta}{8\pi GR^2}(u_a^+ u_b^+ \pm u_a^- u_b^-), \tag{3}$$

where $u^\pm$ are duals to the relatively normalized $[g(u_+, u_-) = -1]$ propagation null vectors,

$$\sqrt{2}u_\pm = \frac{1}{\sqrt{cg_{00}}}\frac{\partial}{\partial Z} \pm \frac{1}{\sqrt{-cg_{11}}}\frac{\partial}{\partial L}. \tag{4}$$

From the Einstein equation the function $\beta$ is found positive, irrespective of the value of $c$,

$$\beta = \frac{R}{ae^{L^2}}. \tag{5}$$

The source can be equally interpreted as an anisotropic fluid with no tangential pressures and both the energy density and radial pressure are equal to $\beta/8\pi GR^2 = 1/8\pi Gae^{L^2}R$ (these also became infinite at $R = 0$). As a third interpretation, it represents a massless scalar field in the two dimensional sector obtained by spherically symmetric reduction of Einstein gravity [6].

The metrics (1) are invariant with respect to the simultaneous interchange $(L, B) \to (-L, -B)$. This suggests the ex-
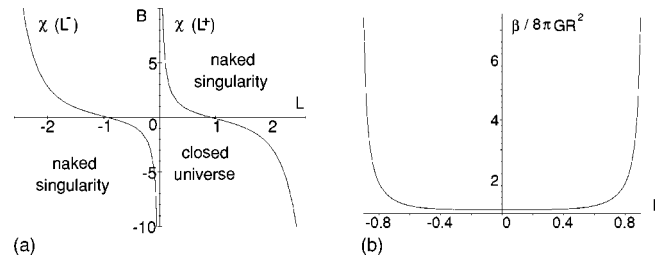


FIG. 1. (a) The singularity $R = 0$ of the metric (1) is represented by the curves $B = \chi(L^\pm)$ which split the range of the coordinate $L$ in three distinct regions: static, homogeneous, and static. (b) Evolution of the energy density of the null dust universe in coordinate time $L$ (in units $a = 8\pi G$).
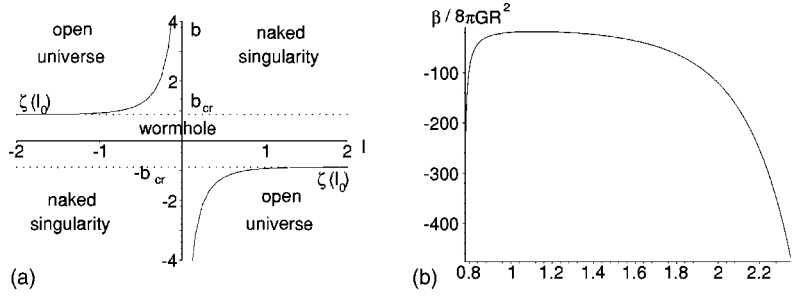
FIG. 2. (a) The singularity $R=0$ of the metric (8) is represented by the curves $b=\zeta(l_0)$. These split the range of the coordinate $l$ in two distinct regions for any $b\in(-\infty,-b_{cr})\cup(b_{cr},\infty)$. For the parameter values $b\in(-b_{cr},b_{cr})$ there is no singularity. (b) Evolution of the energy density of the ghost radiation filled universe in coordinate time $l$ (in units $a=8\pi G$).

istence of two identical copies of the space-time (cf. [7]). For each value of $B$ there are two copies of the singularity $R=0$ which divide the range of the coordinate $L$ in three distinct regions [Fig. 1(a)]. From among these $L\in(-\infty,L^-<0)$ and $L\in(L^+>0,\infty)$ are two identical copies of the static naked singularity given locally by the metric (1), case $c=-1$. The patches $(L^-,0)$ and $(0,L^+)$ can be smoothly glued through $L=0$ in order to obtain a single homogeneous closed Kantowski-Sachs type universe. In this latter case $R$ is interpreted as the time-dependent radius of the universe described locally by the metric (1), case $c=1$, which is born from and collapses to the singularity $R=0$. During its ephemeral existence, the energy density of the universe evolves [cf. Fig. 1(b)].

Solutions describing the crossflow of ghost fields can be found by substituting

$$L=il,\quad Z=i\tau,\quad B=ib \qquad (6)$$

in the metric (1). Then,

$$\Phi_B=i\left(b+\int^l e^{-x^2}dx\right)=i\phi,$$

$$cR=a(e^{-l^2}+2l\phi). \qquad (7)$$

and the new metric becomes

$$ds^2=\frac{a}{e^{l^2}cR}(-d\tau^2+2R^2dl^2)+R^2d\Omega^2. \qquad (8)$$

These are the static solutions of Hayward for $c=1$ ($\tau$ time) and the new homogeneous solutions for $c=-1$ ($l$ time). The signature of the metric (8) being $(-c,c,+,+)$, the propagation null vectors are

$$\sqrt{2}u_\pm=\frac{1}{\sqrt{-cg_{00}}}\frac{\partial}{\partial\tau}\pm\frac{1}{\sqrt{cg_{11}}}\frac{\partial}{\partial l}. \qquad (9)$$

The energy-momentum tensor,

$$T_{ab}=\frac{\beta}{8\pi GR^2}(u_a^+u_b^+\pm u_a^-u_b^-), \qquad (10)$$

contains a *negative* $\beta$, again irrespective of the value of $c$,

$$a\beta=-e^{l^2}R. \qquad (11)$$

Therefore the source represents a crossflow of incoming and outgoing ghost radiation. Alternatively it can be regarded as an anisotropic fluid with no tangential pressures for which both the energy density and radial pressure are equal to $\beta/8\pi GR^2=-e^{l^2}/8\pi GaR<0$ (infinite at $R=0$). In two dimensions it represents a massless scalar field.
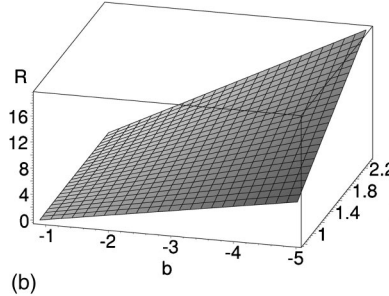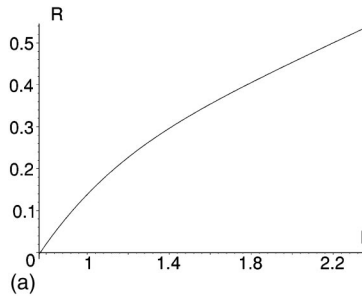


FIG. 3. (a) The homogeneous solution represents an open universe (plot for $b=-1$). (b) The universe expands for any admissible value of $b$.
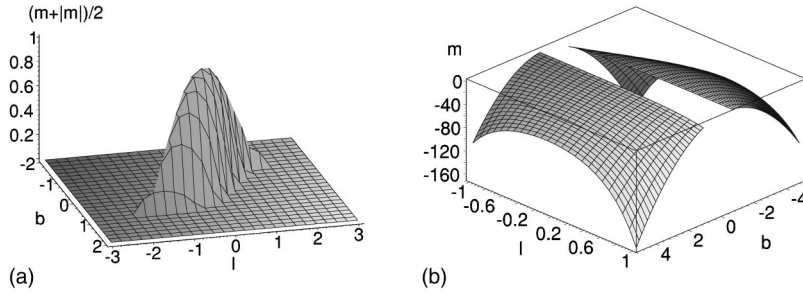
FIG. 4. (a) The mass function is positive only for a restricted domain of $b$, which represents the wormhole in its throat region. (Mass functions are plotted for $a=2$.) (b) The mass function of naked singularities is negative for any $l$.

As $\mathcal{R}_{ab}\mathcal{R}^{ab}=2e^{2l^2}/a^2R^2$ and the Kretschmann scalar $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ still contains $R^6$ in the denominator, the singular character of the solutions at $R=0$ continues to hold. The constant $b$ can be related to the value $l_0$ of the coordinate $l$ at the singularity,

$$b=\zeta(l_0)=-\frac{1}{2l_0e^{l_0^2}}-\frac{1}{2l_0}\int^{l_0}e^{-x^2}dx. \qquad (12)$$

As this equation has no solution for $b\in(-b_{cr},b_{cr})$, with $b_{cr}=0.886\,226\,925\,5$ [Fig. 2(a)], the singularity is absent for the above parameter range, and the corresponding solutions are the wormholes of Hayward. This is the major difference with respect to the null dust case.

For any other value of $b$ the singularity is present. We discuss this possibility next. First we remark the symmetry of the metric with respect to the simultaneous interchange $(l,b)\rightarrow(-l,-b)$. In particular the two branches of the curve $\zeta(l_0)$ represent the same singularity. By inspecting Eq. (7), written in the form

$$cR=2al[\zeta(l)-\zeta(l_0)], \qquad (13)$$

with the remark that $\zeta$ is a monotonously decreasing function $(d\zeta/dl=-e^{-l^2})$, we see that for $b<-b_{cr}$ the singularity divides the $l>0$ coordinate range in two domains, characterized by $c=1$ and $c=-1$, respectively. Similarly for $b>b_{cr}$ the singularity splits $l<0$ in the $c=-1$, $c=1$ domains.

The homogeneous solutions ($c=-1$ case) are new and they lie "outside" the two branches of the curve $\zeta(l_0)$. The evolution of the radius $R$ in coordinate time $l$ is shown in Fig. 3. The solutions represent open universes, either expanding ($b<-b_{cr}$) or contracting ($b>b_{cr}$). This is to be contrasted with the null dust case, where the universe was closed. The evolution of the energy density in the expanding case is shown in Fig. 2(b). Contrarily with open universes filled with ordinary matter, here the magnitude of the energy density first decreases from its original singular value but afterwards it increases again towards infinity.

The region between the two branches of the curve $\zeta(l_0)$ contain the static solutions ($c=1$ case). They are either wormholes or naked singularities. We verify this latter possibility by rewriting the metric (8), case $c=1$ as

$$ds^2=-\frac{a}{e^{-l^2}R}dx^{\pm}(dx^{\pm}-2\sqrt{2}Rdl)+R^2d\Omega^2,$$

where the null coordinate is given by $dx^{\pm}=d\tau\pm\sqrt{2}Rdl$. Radial null geodesics are then characterized by

$$\frac{dl}{dx^{\pm}}=\frac{1}{2\sqrt{2}R}>0, \qquad (14)$$

and there is no horizon.

The mass function for the static solutions [2] is $m=(e^{-l^2}+2l\phi-2e^{2l^2}\phi^2)a/2$. There is a domain of the radial coordinate $l$ with the mass positive, but only for wormholes
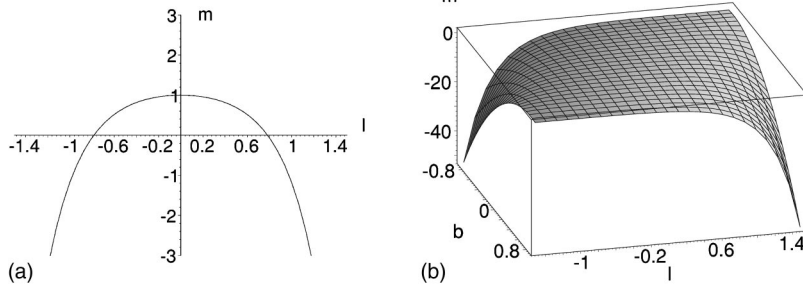


FIG. 5. (a) In the throat region the wormhole has positive mass. (The plot is done for $b=0$.) (b) The missing zone of Fig. 4(b): the mass function for wormholes, represented for various values of the parameter $b$.

[Fig. 4(a)]. Naked singularities are characterized by negative $m$, irrespective of the value of the metric parameters [Fig. 4(b)]. The mass function of static wormholes is represented in Fig. 5. By cutting the wormhole solution at values $l_{\pm}$ (lying in the positive mass range), and gluing to spherically symmetric exterior solutions (as described in [1]) we obtain a wormhole composed of a crossflow of negative energy radiation, still with positive mass.

Summarizing, by switching from a positive to a negative energy density (ghost radiation), the static naked singularity is either kept or it transmutes to a (possibly positive mass) wormhole. In addition, the homogeneous closed universe opens up.

A final remark is that all solutions considered here exhibit a vanishing Ricci scalar, similarly as the wormhole solutions of [3].

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